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QUALITATIVE INVESTIGATION OF THE DYNAMICS OF A TOP ON A PLANE WITH FRICTION*

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Study of the case of a bifurcation manifold and domains of possible motions in the problem of a top spinning, with slipping, on a horizontal plane with friction, is used as the basis when considering the problem of generalization and corresponding modification of the Smiles theory /1, 2/ to a dissipative system with symmetry.

The equations of motion of a heavy, inhomogeneous, dynamically symmetric sphere on a horizontal plane with slippage friction admit of the following function which does not increase along all motions of the sphere:

$$H = {}^1/_2 m(v_1^2 + v_2^2 + v_3^2) + {}^1/_2 J_1(\omega_1^2 + \omega_2^2) + {}^1/_2 J_3\omega_3^2 - mga\gamma_3 \leqslant h$$

and two integrals

$$K = J_1(\omega_1\gamma_1 + \omega_2\gamma_2) + J_3\omega_3(\gamma_3 - a/r) = k$$
$$\Gamma = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

Here m is the mass of the top, J_1 and J_3 is the equatorial and axial central moments of inertia, respectively, g is the acceleration due to gravity, a is the displacement of the geometrical centre of the sphere from its centre of mass along the dynamic axis of symmetry (the positive direction of this axis is chosen so that $a > 0$), r is the radius of the sphere, v_i, ω_i and γ_i $(i = 1, 2, 3)$ are the projections of the velocity vectors $\mathbf{v}$ of the centre of mass of the top, of its angular velocity ω and of the unit vector γ of the ascending vertical respectively, on the principal central axes of inertia of the top.

Let $W_k(\gamma)$ be the minimum of the function $H(\mathbf{v}, \omega, \gamma)$ with respect to the variable $\mathbf{v}$ and ω at the level k of the Gellet integral $K = k$:

$$W_k(\gamma) = \frac{k^2}{2[J_1(\gamma_1^2 + \gamma_2^2) + J_3(\gamma_3 - \varepsilon)^2]} - mga\gamma_3 \quad (\varepsilon = a/r < 1)$$

It is clear that the following assertions hold:

1°. The function $W_k(\gamma)$ on the sphere $S^2 = \{\gamma: \gamma^2 = 1\}$ takes stationary values if and only if the function H takes stationary values at constant values of the integrals $K = k$ and $\Gamma = 1$.

2°. The function $W_k(\gamma)$ on the sphere S^2 reaches a strict minimum (maximum) value if and only if the function H reaches a strict minimum (does not reach a weak minimum) at constant values of the integrals $K = k$ and $\Gamma = 1$.

As we know /3, 4/, the critical values of the function H correspond, for fixed constant integrals $K = k$ and $\Gamma = 1$, to invariant manifolds of the system in question. Moreover, if the frictional force vanishes at zero velocity of slippage, the function H will retain its initial value only in these manifolds and will decrease on all remaining motions of the top. Therefore the following assertion holds.

3°. The critical values of the function $W_k(\gamma)$ on the sphere S^2 define the bifurcation manifold

$$\Sigma = \{(k, W_k(\mathbf{M}), \mathbf{M} \in S^2 : \delta W_k(\mathbf{M}) = 0\}$$

on passing through which the type of domains of possible motion

$$\mathbf{M}_{h,k} = \{\gamma \in S^2, W_k(\gamma) \leqslant h\}$$

changes, and to points of the minimum of the function $W_k(\gamma)$ on the sphere S^2 there will correspond stable (asymptotically with respect to a part of the variables) and to points of the maximum, unstable invariant manifolds of the system in question.

We note that the assertions of stability or instability of invariant manifolds follow from the corresponding modifications of Routh's theorem /5/.

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It is clear that the manifold M on which the function $W_k(\gamma)$ on the sphere S^2 takes critical values, always contains two points $P_\pm$ $(\gamma_1 = \gamma_2 = 0, \gamma_3 = \pm 1)$ which are the poles of Poisson's sphere to which the rotation of the top about the vertically placed dynamic symmetry axis correspond. Moreover, the manifold may contain not more than two circles S_γ $(\gamma_3 = \gamma;\ \gamma_1^2 + \gamma_2^2 = 1 - \gamma^2)$ where γ is the real root of the equation

$$\frac{dV_k}{d\gamma_3} = 0 \quad \left(V_k = \frac{k^2}{2\left[J_1(1-\gamma_3^2) + J_3(\gamma_3 - \varepsilon)^2\right]} - mga\,\gamma_3\right)$$

smaller than unity in modulo (there cannot be more than two such roots /3, 4/). The circles S_γ are the parallels of Poisson's sphere and regular precessions of the top correspond to them.

Thus the bifurcation manifold Σ represents, in the problem discussed here, a union of two manifolds $\Sigma_\pm$ and Σ_γ corresponding to the critical points of the function $W_k(\gamma)$ of the form $\gamma_3 = \pm 1$, $\gamma_3 = \gamma$.

In the space $(k;\ h)$ (the constant of the Gellet integral, the value of the total energy) the sets $\Sigma_\pm$ represent parabolas

$$h = \frac{k^2}{2J_3(1 \pm \varepsilon)^2} \mp mga$$

and the set Σ_γ represents a curve whose parametric representation has the form (p is a parameter)

$$h = \frac{mga}{2(1-\delta)p}\left[\delta(1 - \varepsilon^2 - \delta)p^2 - \varepsilon p + 3\right]$$

$$k^2 = \frac{J_3\, mga}{(1-\delta)^2 p^3}\left[\delta(1 - \varepsilon^2 - \delta)p^2 + 1\right]$$

$$(\delta = J_1/J_3 > 1/2)$$

The existence of the set Σ_γ, its form, and the character of the extremum of the function $W_k(\gamma)$ on the set Σ depend on the parameters $\varepsilon = a/r \in (0;\ 1)$ and $\delta = J_1/J_3 > 1/2$ and on the constant k of the Gellet integral. This requires that the following six cases be considered:

$$\delta > 1 + \varepsilon;\ 0 < \varepsilon < 1 \quad (p > p_1) \tag{1}$$

$$1 + \varepsilon > \varphi_+(\varepsilon)\ (\delta \neq 1);\ 0 < \varepsilon < 1 \quad (p \in (p_1;\ p_2)) \tag{2}$$

$$\varphi_+(\varepsilon) > \delta > \begin{cases} 1 - \varepsilon, & 1 > \varepsilon > \varepsilon_0 \\ \varphi_+(-\varepsilon), & 0 < \varepsilon < \varepsilon_0 \end{cases} \quad (p \in (p_2;\ p_1)) \tag{3}$$

$$\delta < \begin{cases} 1 - \varepsilon, & 1 > \varepsilon > \varepsilon_0 \\ \varphi_-(-\varepsilon), & 0 < \varepsilon < \varepsilon_0 \end{cases} \quad (p > p_2) \tag{4}$$

$$\varphi_+(-\varepsilon) > \delta > 1 - \varepsilon,\ 0 < \varepsilon < \varepsilon_0 \quad (p \in (p_2;\ p_1)) \tag{5}$$

$$1 - \varepsilon > \delta > \varphi_-(-\varepsilon),\ 0 < \varepsilon < \varepsilon_0 \quad (p > p_2) \tag{6}$$

Here (and henceforth) the following notation is adopted:

$$p_1 = \frac{1}{\delta + \varepsilon - 1},\quad p_2 = \frac{1}{1 + \varepsilon - \delta};\quad p_0^2 = \frac{3}{\delta(1 - \varepsilon^2 - \delta)}$$

$$\varphi_\pm(\varepsilon) = \frac{1-\varepsilon}{8}\left(7 + \varepsilon \pm \sqrt{1 + 14\varepsilon + \varepsilon^2}\right),\quad \varepsilon_0 = 7 - \sqrt{48}$$

$$k_0 = k(p_0);\ k_1 = k(p_1);\ k_2 = k(p_2);\ k_* = (1 - \varepsilon^2)\sqrt{J_3 mga/\varepsilon}$$

The cases (1)-(6) correspond to the bifurcation diagrams in the corresponding figures, and thick lines are used to depict the segments of the bifurcation set Σ corresponding to the minimum of the effective potential $W_k(\gamma)$.

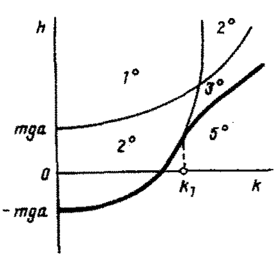

Fig.1

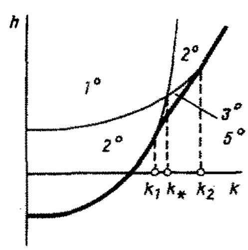

Fig.2

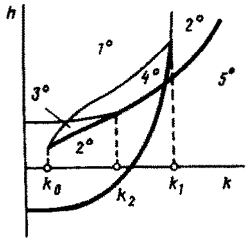

Fig.3

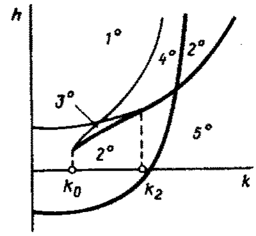

Fig.4

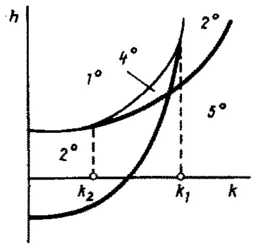

Fig.5

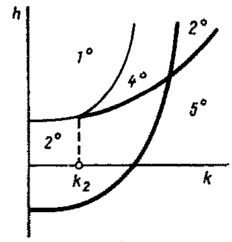

Fig.6

We note that the representative point of the plane $(k; h)$ belonging to the bifurcation set Σ is invariant with respect to the phase flux of the system in question. Any other point not belonging to the set Σ will move along the straight line $k = \text{const}$ in the direction of decreasing h, asymptotically approaching the nearest point from below of the set Σ, lying on this straight line.

The domains of possible motions will change topologically during the passage throught the bifurcation manifold: numbers 1°-5° in the figures denote the domains of possible motion of the type S^2 (1°), D^2 (2°); $S^1 \times D^1$ (3°); $D^2 U D^2$ (4°); ϕ (5°).

The boundary of region 5° (the empty set) corresponds to the global minimum of the function $W_k(\gamma)$ on the sphere S^2. When the values of the total energy of the top h and the constant of the Gellet integral k lie below this boundary, motion is impossible.

The results given here complement the qualitative study of a top on a plane with friction /5, 6/ and makes it possible, in particular, to determine uniquely the final motions of the top for the given values of its parameters. For example, if ε and δ satisfy conditions (2) (see Fig.2), then for the values $(k; h)$ lying within the region 2°, $k < k_*$ the final motions will be rotations about the vertical, with the centre of mass at its lowest position. When $(k; h)$ lie within the region 3°, the final motions will be regular precessions, etc.

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